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Cole–Hopf-like transformation for Schrödinger equations containing complex nonlinearities

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Abstract

We consider systems which conserve the particle number and are described by Schrödinger equations containing complex nonlinearities. In the case of canonical systems, we study their main symmetries and conservation laws. We introduce a Cole–Hopf-like transformation both for canonical and noncanonical systems, which changes the evolution equation into another one containing purely real nonlinearities, and reduces the continuity equation to the standard form of the linear theory. This approach allows us to treat, in a unifying scheme, a wide variety of canonical and noncanonical nonlinear systems, some of them already known in the literature.

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1. Introduction

Over the last few decades many nonlinear Schrödinger equations (NLSEs) have been proposed in order to test the fundamental postulates of quantum mechanics, for instance, the Bialynicki-Birula–Mycielski equation [1], the Kostin equation [2] and the Gisin equation [3] among many others [4]. In [5] a wide class of NLSEs for finite-dimensional quantum systems was selected in order to preserve the *homogeneity principle* of the original Schrödinger equation, with the *superposition principle* being destroyed by the nonlinear terms.

Many of the NLSEs proposed in the literature contain complex nonlinearities. For instance, the Doebner–Goldin (DG) equations [6–8] were introduced as the most general class of Schrödinger equations, compatible with the Fokker–Planck equation for the probability density $\rho = |\psi|^2$ namely $\partial\rho/\partial t + \nabla \cdot \boldsymbol{j}_0 = D\Delta\rho$, $\boldsymbol{j}_0$ being the standard quantum current and D a positive diffusion coefficient. The importance of this class of evolution equations is that it is founded on the grounds of the group theory: the nonlinear terms were derived from the representation analysis of the Diff$(\boldsymbol{R}^3)$ group which was proposed as a *universal quantum kinematical group* [9].

In addition, a large number of NLSEs with complex nonlinearities have been proposed in order to describe some phenomenologies in condensed matter physics. For instance, in [10] a NLSE with a nonlinearity of the type $a_1|\psi|^2\psi + a_2|\psi|^4\psi + \mathrm{i}a_3\partial_x(|\psi|^2\psi) + (a_4 + \mathrm{i}a_5)\partial_x|\psi|^2\psi$ is introduced to describe a single-mode wave propagation in a Kerr dielectric guide. Another example is the generalized Ginsburg–Landau equation [11]. This equation contains the nonlinearity $a_1|\psi|^2\psi + \mathrm{i}a_2\psi + \mathrm{i}a_3\partial_{xx}\psi + \mathrm{i}a_4|\psi|^2\psi$ [12] which takes into account pumping and dumping effects of the nonlinear media and can be used to describe dynamical modes of plasma physics, hydrodynamics and also solitons in optical fibres (see [13] and references therein). Finally, complex nonlinearities in Schrödinger equations are also used to describe the propagation of high-power optical pulses in ultrashort soliton communication systems [14, 15], incoherent solitons [16, 17], and multi-channel bit-parallel-wavelength optical fibre networks [18], among others.

In this paper we consider the most general class of NLSEs conserving the quantity $N = \int |\psi|^2\,\mathrm{d}^n x$:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + (W + \mathrm{i}\mathcal{W})\psi \tag{1.1}$$

where the real W and imaginary $\mathcal{W}$ parts of the complex nonlinearity are smooth functions of the fields ψ, ψ^* and their spatial derivatives of any order. When ψ is written in polar representation $\psi = \rho^{1/2}\exp(\mathrm{i}S/\hbar)$, equation (1.1) is split into two nonlinear partial differential equations for the real fields ρ and S:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}S}{m}\rho + \boldsymbol{F}\right) = 0 \tag{1.2}$$

$$\frac{\partial S}{\partial t} + \frac{(\boldsymbol{\nabla}S)^2}{2m} + W + U_q = 0 \tag{1.3}$$

where

$$U_q = -\frac{\hbar^2}{4m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{1.4}$$

is the quantum potential [19] and the real functional $\boldsymbol{F}$ is related to $\mathcal{W}$ through

$$\mathcal{W} = \frac{\hbar}{2\rho}\boldsymbol{\nabla}\cdot\boldsymbol{F} \tag{1.5}$$

as the particle number conservation requires. It is easy to recognize that equation (1.2) is a nonlinear continuity equation, which involves only the term $\mathcal{W}$, while equation (1.3) is a nonlinear *Hamilton–Jacobi*-like equation involving only the term W.

In the Calogero picture [20], the system (1.2), (1.3) is C-integrable if there exists a transformation of the dependent or/and independent variables $t \to T$, $\boldsymbol{x} \to \boldsymbol{X}$, $\rho \to R$, $S \to \mathcal{S}$ which transforms equations (1.2), (1.3) into

$$\frac{\partial R}{\partial T} + \overline{\boldsymbol{\nabla}}\cdot\left(\frac{\overline{\boldsymbol{\nabla}}\mathcal{S}}{m}R\right) = 0 \tag{1.6}$$

$$\frac{\partial\mathcal{S}}{\partial T} + \frac{(\overline{\boldsymbol{\nabla}}\mathcal{S})^2}{2m} + \overline{U}_q = 0 \tag{1.7}$$

$\overline{\boldsymbol{\nabla}}$ and $\overline{U}_q$ being the gradient and the quantum potential in the new variables. Equations (1.6), (1.7) constitute the well known hydrodynamic representation of the standard linear Schrödinger equation.

The principal aim of this paper is to introduce a nonlinear transformation for the field S: $S \to \mathcal{S}$, in order to reduce equation (1.2) to the standard form of the linear theory (1.6). As a consequence of this transformation, the evolution equation (1.1) transforms into another one containing a purely real nonlinearity. Moreover, the current, when expressed in terms of the new field $\phi = \rho^{1/2}\exp(\mathrm{i}\mathcal{S}/\hbar)$, reduces to the standard bilinear form of the linear Schrödinger theory.

The paper is organized as follows. In section 2, we introduce a general class of $(n+1)$ canonical NLSEs, invariant over the action of the $U(1)$ group. In section 3, starting from the Noether theorem, we consider the main symmetries and related conserved quantities of the canonical system. In section 4, we introduce a Cole–Hopf-like transformation which eliminates the imaginary part of the nonlinearity in the evolution equation, while in section 5, the same transformation is considered in the case of noncanonical systems. In section 6, in the framework of the approach developed in the previous sections, we treat, in a unifying context, some NLSEs already known in the literature, in order to show that all the transformations introduced by the various authors to study these equations can be obtained as particular cases of the transformation proposed here. Finally, some conclusions and remarks are reported in section 7.

2. The canonical model

Let us consider the class of canonical NLSEs described by the Lagrangian density

$$\mathcal{L} = \mathrm{i}\frac{\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2 - U[\psi^*, \psi] \tag{2.1}$$

where $\boldsymbol{\nabla} \equiv (\partial_1, \ldots, \partial_n)$ is the n-dimensional gradient operator. The last term in the rhs of equation (2.1) is the nonlinear potential which we assume to be a real smooth function of the fields ψ and ψ^* and their spatial derivatives. Here and in the following, we use the notation $U[a]$ to indicate the dependence of U on the field a and its spatial derivative of any order. We deal with dynamical systems described by equation (2.1) which are invariant under transformations belonging to the $U(1)$ group. As we show in the next section, this condition imposes a constraint on the form of the nonlinear potential U.

We start from the action

$$\mathcal{A} = \int \mathcal{L}\, \mathrm{d}^n x\, \mathrm{d}t \tag{2.2}$$

and observe that the evolution equation of the field ψ is given by

$$\frac{\delta\mathcal{A}}{\delta\psi^*} = 0. \tag{2.3}$$

The functional derivative is defined through [21]

$$\frac{\delta}{\delta a}\int \mathcal{G}[a]\,\mathrm{d}^n x = \sum_{[k=0]}(-1)^k\mathcal{D}_{I_k}\left[\frac{\partial\mathcal{G}[a]}{\partial(\mathcal{D}_{I_k}a)}\right] \tag{2.4}$$

with $\mathcal{D}_{I_k} \equiv \partial^k/(\partial x_1^{i_1}\ldots x_n^{i_n})$ and $\sum_{[k=0]} \equiv \sum_{k=0}^{\infty}\sum_{I_k}$. The sum $\sum_{I_k}$ is over the multi-index $I_k \equiv (i_1, i_2, \ldots, i_n)$ where $1 \leqslant p \leqslant n$, $0 \leqslant i_p \leqslant k$ and $\sum i_p = k$.

Equation (2.3) assumes the form

$$\begin{aligned}\frac{\delta}{\delta\psi^*}\int \mathrm{i}\frac{\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\mathrm{d}^n x\,\mathrm{d}t\\ = \frac{\delta}{\delta\psi^*}\int\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2\,\mathrm{d}^n x\,\mathrm{d}t + \frac{\delta}{\delta\psi^*}\int U[\psi^*,\psi]\,\mathrm{d}^n x\,\mathrm{d}t\end{aligned} \tag{2.5}$$

which, after performing the functional derivatives, transforms to the following NLSE:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \frac{\delta}{\delta\psi^*}\int U[\psi^*, \psi]\,\mathrm{d}^n x\,\mathrm{d}t \tag{2.6}$$

where $\Delta \equiv \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian operator. Equation (2.6) can finally be written in the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \Lambda[\rho, S]\psi \tag{2.7}$$

where the complex nonlinearity

$$\Lambda[\rho, S] = W[\rho, S] + \mathrm{i}\mathcal{W}[\rho, S] \tag{2.8}$$

has real $W[\rho, S]$ and imaginary $\mathcal{W}[\rho, S]$ part defined by

$$W[\rho, S] = \frac{\delta}{\delta\rho}\int U[\rho, S]\,\mathrm{d}^n x\,\mathrm{d}t \tag{2.9}$$

$$\mathcal{W}[\rho, S] = \frac{\hbar}{2\rho}\frac{\delta}{\delta S}\int U[\rho, S]\,\mathrm{d}^n x\,\mathrm{d}t \tag{2.10}$$

where ρ and S are the hydrodynamic fields related to the wavefunction ψ through [19, 22]

$$\psi(\boldsymbol{x}, t) = \rho^{1/2}(\boldsymbol{x}, t)\exp\left[\frac{\mathrm{i}}{\hbar}S(\boldsymbol{x}, t)\right]. \tag{2.11}$$

3. Symmetries

In this section we study the main symmetries and conserved quantities of the system described by the Lagrangian (2.1).

Let us consider the $U(1)$ invariance condition. The variation $\delta_\epsilon\psi = \mathrm{i}\epsilon\psi$, with ϵ an infinitesimal real parameter, implies the following variation on the action:

$$\delta_\epsilon\mathcal{A} = -\epsilon\hbar\int\frac{\partial}{\partial S}U[\rho, S]\,\mathrm{d}^n x\,\mathrm{d}t. \tag{3.1}$$

Taking into account the Noether theorem [23] we can also write the variation $\delta_\epsilon\mathcal{A}$ in the form

$$\delta_\epsilon\mathcal{A} = -\epsilon\hbar\int\partial_\mu j_\mu[\psi^*, \psi]\,\mathrm{d}^n x\,\mathrm{d}t. \tag{3.2}$$

By comparing equations (3.1) and (3.2) we obtain

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = \frac{\partial U}{\partial S} \tag{3.3}$$

with $\rho = j_0$. Equation (3.3) is not a continuity equation because the Lagrangian (2.1), for a general nonlinear potential $U[\rho, S]$, is not $U(1)$-invariant. In appendix B we show that $U(1)$ symmetry can be restored if one assumes that the nonlinear potential $U[\rho, S]$ depends on S only through its spatial derivative, modulo a total derivative term, which does not change the dynamics of the system (null Lagrangian). As a consequence, the rhs of equation (3.3) vanishes and it becomes a continuity equation for the conserved density ρ. Thus, the $U(1)$ invariance limits the class of nonlinear potentials appearing in equation (2.1). In the following, we consider only $U(1)$-invariant systems, where the functional $U[\rho, S]$ depends on S through its spatial derivative. For this class of systems, equation (3.3) becomes

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0. \tag{3.4}$$

The conserved quantity associated with the continuity equation (3.4) is

$$N = \int \rho \, \mathrm{d}^n x \tag{3.5}$$

The expression of $\boldsymbol{j}$ is obtained in appendix A, and is given by

$$j_i = \frac{\partial_i S}{m}\rho + \sum_{[k=0]} \frac{(-1)^k}{f_i^{I_{k+1}}} \mathcal{D}_{I_k}\left[\frac{\partial U[\rho, S]}{\partial(\mathcal{D}_{i,I_k} S)}\right] \tag{3.6}$$

where

$$f_i^{I_{k+1}} = n - \sum_{r \neq i}^{n} \delta_{0,m_r} \tag{3.7}$$

for $I_{k+1} = (m_1, m_2, \ldots, m_n)$ with $1 \leqslant r \leqslant n$, $0 \leqslant m_r \leqslant n$ and $\sum_r m_r = k + 1$.

Note that expression (3.6) of $\boldsymbol{j}$ can also be written (see appendix A) in the form

$$j_i = \frac{\partial_i S}{m}\rho + \frac{\delta}{\delta(\partial_i S)} \int U[\rho, S] \, \mathrm{d}^n x \, \mathrm{d}t. \tag{3.8}$$

Equation (3.8) can be obtained starting directly from equation (2.7) after adopting the hypothesis that $U[\rho, S]$ depends on the field S only through its spatial derivatives as required from the $U(1)$ symmetry.

In the following we consider the main space–time symmetries of the Lagrangian (2.1). We note that $U[\rho, S]$ depends on the variables $\boldsymbol{x}$ and t only through the fields ρ and S, thus the system is invariant over space–time translations. From the Noether theorem we have

$$\frac{\partial \mathcal{T}_\mu}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{T}}_\mu = 0 \tag{3.9}$$

where $\mathcal{T}_\mu \equiv T_{0\mu}$; $(\boldsymbol{\mathcal{T}}_\mu)_i \equiv T_{i\mu}$ with $\mu = 0, \ldots, 3$. The components of the energy–momentum tensor $T_{\mu\nu}$ (see appendix A) are given by

$$T_{00} = \frac{\hbar^2}{2m} |\boldsymbol{\nabla}\psi|^2 + U[\rho, S] \tag{3.10}$$

$$T_{0j} = \mathrm{i}\frac{\hbar}{2}(\psi^* \partial_j \psi - \psi \partial_j \psi^*) \tag{3.11}$$

$$\begin{aligned} T_{i0} = &-\frac{\hbar^2}{2m}(\partial_i \psi^* \partial_t \psi - \partial_i \psi \partial_t \psi^*) + \sum_{[k=0]} \sum_{[p=0]}^{k} (-1)^p B_{j,I_q}^{I_k} \mathcal{D}_{I_q} \\ &\times \left\{ \mathcal{D}_{I_p}\left[\frac{\partial U[\rho, S]}{\partial(\mathcal{D}_{i,I_k}\rho)}\right] \partial_t \rho + \mathcal{D}_{I_p}\left[\frac{\partial U[\rho, S]}{\partial(\mathcal{D}_{i,I_k} S)}\right] \partial_t S \right\} \end{aligned} \tag{3.12}$$

$$\begin{aligned} T_{ij} = &-\frac{\hbar^2}{2m}(\partial_i \psi^* \partial_j \psi + \partial_j \psi^* \partial_i \psi) + \delta_{ij}\mathcal{L} - \sum_{[k=0]} \sum_{[p=0]}^{k} (-1)^p B_{j,I_q}^{I_k} \mathcal{D}_{I_q} \\ &\times \left\{ \mathcal{D}_{I_p}\left[\frac{\partial U[\rho, S]}{\partial(\mathcal{D}_{i,I_k}\rho)}\right] \partial_j \rho + \mathcal{D}_{I_p}\left[\frac{\partial U[\rho, S]}{\partial(\mathcal{D}_{i,I_k} S)}\right] \partial_j S \right\}. \end{aligned} \tag{3.13}$$

From equations (3.10) and (3.11) we obtain the conserved quantities

$$E = \int \left[\frac{\hbar^2}{2m} |\boldsymbol{\nabla}\psi|^2 + U[\rho, S]\right] \mathrm{d}^n x \tag{3.14}$$

$$\boldsymbol{P} = -\mathrm{i}\frac{\hbar}{2} \int (\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*) \, \mathrm{d}^n x \tag{3.15}$$

which are, respectively, the total energy and the linear momentum of the system. We can see that $U[\rho, S]$ modifies the expression of the energy while the momentum maintains the form of the linear theory. From equations (3.12) and (3.13) we see that the presence of $U[\rho, S]$ also modifies the expressions of the fluxes associated to E and $\boldsymbol{P}$.

Note that if the energy–momentum tensor is symmetric in the spatial indices $T_{ij} = T_{ji}$, the potential $U[\rho, S]$ is invariant over the action of the orthogonal group $SO(n)$. In this way we can define $n(n-1)/2$ conserved quantities

$$L^{a_1,\ldots,a_{n-2}} = \epsilon^{a_1,\ldots,a_{n-2},i,j} \int x_i T_{0j}\, \mathrm{d}^n x \tag{3.16}$$

where $\epsilon^{a_1,\ldots,j}$ is the n-rank totally antisymmetric tensor defined as $\epsilon^{1,\ldots,1} = 1$. For $n = 3$ we recognize the well known conserved components of the angular momentum.

Finally, we look at the Galilei invariance. We recall that if the system admits this symmetry, the corresponding generator

$$\boldsymbol{G} = \boldsymbol{P}t - mN\boldsymbol{x}_c \tag{3.17}$$

is conserved, namely

$$\frac{\partial \boldsymbol{G}}{\partial t} = 0 \tag{3.18}$$

where the linear momentum is given by

$$\boldsymbol{P} = \int \rho \boldsymbol{\nabla} S\, \mathrm{d}^n x \tag{3.19}$$

while the mass centre vector is defined as

$$\boldsymbol{x}_c = \frac{1}{N} \int \rho \boldsymbol{x}\, \mathrm{d}^n x. \tag{3.20}$$

We consider now the Ehrenfest relation

$$\frac{\partial \boldsymbol{x}_c}{\partial t} = \frac{1}{N} \int \boldsymbol{j}\, \mathrm{d}^n x \tag{3.21}$$

which can be obtained from equations (3.4) and (3.20). We note the formal similarity with the corresponding relation of the linear theory. Here the expression of $\boldsymbol{j}$ is given by equation (3.6) and depends on the form of the nonlinear potential $U[\rho, S]$. Taking into account the conservation of $\boldsymbol{P}$, and after assuming uniform conditions, from equation (3.17) we obtain

$$\frac{\partial G^i}{\partial t} = -m \sum_{[k=0]} (-1)^k (f_i^{I_k})^{-1} \int \mathcal{D}_{I_k} \left[\frac{\partial U[\rho, S]}{\partial(\mathcal{D}_{i,I_k} S)} \right] \mathrm{d}^n x. \tag{3.22}$$

Equation (3.22) shows that the presence of $U[\rho, S]$ breaks the Galilei invariance, which can be restored if $\mathcal{W} = 0$ as we can verify easily by using equation (2.10).

4. A Cole–Hopf-like transformation

Let us introduce a unitary transformation of the field ψ:

$$\psi(\boldsymbol{x}, t) \to \phi(\boldsymbol{x}, t) = \mathcal{U}[\rho, S]\psi(\boldsymbol{x}, t) \tag{4.1}$$

with $\mathcal{U}^* = \mathcal{U}^{-1}$ so that

$$|\psi|^2 = |\phi|^2 = \rho. \tag{4.2}$$

The functional $\mathcal{U}$ is chosen to eliminate the imaginary part of the NLSE (2.7) and, at the same time, to transform the current $\boldsymbol{j}$, given by equation (3.6), into another current $\boldsymbol{j} \to \boldsymbol{J}$ having the canonical form

$$\boldsymbol{J} = \frac{\boldsymbol{\nabla} \mathcal{S}}{m} \rho \tag{4.3}$$

with $\mathcal{S}$ being the phase of the new field ϕ:

$$\phi(\boldsymbol{x}, t) = \rho^{1/2}(\boldsymbol{x}, t) \exp\left[\frac{\mathrm{i}}{\hbar}\mathcal{S}(\boldsymbol{x}, t)\right]. \tag{4.4}$$

We write $\mathcal{U}[\rho, S]$ as follows:

$$\mathcal{U}[\rho, S] = \exp\left(\frac{\mathrm{i}}{\hbar}\sigma[\rho, S]\right) \tag{4.5}$$

and observe that the generator $\sigma[\rho, S]$ is a real functional which allows us to calculate $\mathcal{S}$ starting from S:

$$\mathcal{S} = S + \sigma[\rho, S]. \tag{4.6}$$

The generator $\sigma[\rho, S]$ can be obtained easily by combining equations (3.8), (4.3) and (4.6):

$$\partial_i \sigma[\rho, S] = \frac{m}{\rho}\frac{\delta}{\delta(\partial_i S)}\int U[\rho, S]\,\mathrm{d}^n x\,\mathrm{d}t. \tag{4.7}$$

Equation (4.7) imposes a condition on the form of the nonlinear potential which can be obtained using the relation $\partial_{ij}\sigma = \partial_{ji}\sigma$:

$$\left[\partial_i\left(\frac{1}{\rho}\frac{\delta}{\delta(\partial_j S)}\right) - \partial_j\left(\frac{1}{\rho}\frac{\delta}{\delta(\partial_i S)}\right)\right]\int U[\rho, S]\,\mathrm{d}^n x\,\mathrm{d}t = 0. \tag{4.8}$$

Condition (4.8) selects the potentials $U[\rho, S]$ and the nonlinear systems where we can perform the transformation (4.1). In the case of one-dimensional systems the transformation (4.1) is always accomplished.

It is easy to verify that the transformation (4.1) reduces the evolution equation (2.7) to the following NLSE:

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \tilde{W}[\rho, \mathcal{S}]\phi \tag{4.9}$$

which now contains only the real nonlinearity $\tilde{W}[\rho, \mathcal{S}]$ given by

$$\tilde{W}[\rho, \mathcal{S}] = W + \frac{(\boldsymbol{\nabla}\sigma)^2}{2m} - \frac{\boldsymbol{J}\cdot\boldsymbol{\nabla}\sigma}{\rho} - \frac{\partial\sigma}{\partial t} \tag{4.10}$$

where $W \equiv W[\rho, S[\rho, \mathcal{S}]]$. The phase $\mathcal{S}$ appears in equation (4.9) only through its spatial derivatives; consequently the arbitrary integration constant, deriving from the definition of $\mathcal{U}$, does not produce effects and can be posed equal to zero. Note that $\tilde{W}$ depends implicitly on the field $\mathcal{S}$. In fact, equation (4.6) defines S as a function of ρ and $\mathcal{S}$.

From equation (4.9) we can obtain the following continuity equation:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \tag{4.11}$$

where the current $\boldsymbol{J}$ now takes the standard expression of the linear quantum mechanics given by equation (4.3).

In conclusion, we have introduced a nonlinear and nonlocal transformation which makes real the complex nonlinearity in equation (2.7) and at the same time reduces the continuity equation (3.3) to the bilinear standard form. The price that we pay is that equation (4.9) is not generally canonical because the transformation (4.1) is itself not canonical.

We briefly discuss the conditions under which the system described by equation (4.9) becomes a canonical one. The canonicity of the system implies the existence of a nonlinear potential $\tilde{U}$ from which we can derive the nonlinearity of equation (4.9).

We observe that the absence of the imaginary part $\tilde{\mathcal{W}}$ in the nonlinearity of equation (4.9) requires that $\tilde{U}$ depends only on the field ρ and its spatial derivatives. Consequently, $\tilde{W}$ is a functional of the field ρ linked with $\tilde{U}(\rho)$ through the relation

$$\tilde{W}[\rho] = \frac{\delta}{\delta\rho}\int \tilde{U}[\rho]\,\mathrm{d}^n x\,\mathrm{d}t \tag{4.12}$$

which, after performing the functional derivative, assumes the form

$$\tilde{W}[\rho] = \sum_{[k=0]}(-1)^k \mathcal{D}_{I_k}\left[\frac{\partial\tilde{U}[\rho]}{\partial(\mathcal{D}_{I_k}\rho)}\right]. \tag{4.13}$$

In section 6 we consider a few particular systems where condition (4.13) is satisfied and their canonicity is preserved.

5. Noncanonical systems

In this section we consider the transformation introduced previously and study its applicability in the case of noncanonical systems.

For noncanonical systems, the evolution equation is given by equation (1.1), where $W[\rho, S]$ is now an arbitrary functional, while $\mathcal{W}[\rho, S]$ assumes the form

$$\mathcal{W}[\rho, S] = \frac{\hbar}{2\rho}\boldsymbol{\nabla}\cdot\boldsymbol{F}[\rho, S] \tag{5.1}$$

enforced by the conservation of $N = \int \rho\,\mathrm{d}^n x$, with $\boldsymbol{F}[\rho, S]$ an arbitrary functional, and the current is given by

$$\boldsymbol{j} = \frac{\boldsymbol{\nabla}S}{m}\rho - \boldsymbol{F}[\rho, S]. \tag{5.2}$$

It is easy to verify that the transformation (4.1) eliminates the imaginary part of the nonlinearity in the motion equation, which transforms again into equations (4.9), (4.10). The generator σ of the transformation $\psi \to \phi$ is related to $\boldsymbol{F}$ through

$$\boldsymbol{\nabla}\sigma[\rho, S] = \frac{m}{\rho}\boldsymbol{F}[\rho, S] \tag{5.3}$$

while the condition

$$\boldsymbol{\nabla}\times\frac{\boldsymbol{F}}{\rho} = 0 \tag{5.4}$$

permits the definition of the transformation in any $n > 1$ spatial dimension. This condition constrains only the form of $\mathcal{W}$, differently to the canonical case, where the condition (4.8) constrains the form of the nonlinear potential U and, consequently, both W and $\mathcal{W}$ in the motion equation.

6. Examples

In this section we consider some equations already known in the literature in the framework of the approach developed here. We show how the nonlinear transformations proposed to study the various NLSEs can be obtained in a unified way as particular cases of the transformation given by equation (4.1).

(i) As a first trivial example we consider the canonical NLSE introduced in [16]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \left(\beta\rho - \frac{\boldsymbol{\alpha}}{\hbar}\cdot\boldsymbol{\nabla}S\right)\psi + \frac{i}{2}(\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\log\rho)\psi \tag{6.1}$$

with β and $\boldsymbol{\alpha}$ real and constant arbitrary parameters. This equation can be derived from a Lagrangian containing the following nonlinear potential:

$$U[\rho, S] = \frac{\beta}{2}\rho^2 - \frac{\rho}{\hbar}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}S. \tag{6.2}$$

The transformation with generator σ, given by

$$\sigma = -\frac{m}{\hbar}\boldsymbol{\alpha}\cdot\boldsymbol{x} \tag{6.3}$$

produces the new canonical evolution equation

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \beta\rho\phi + \frac{m\boldsymbol{\alpha}^2}{2\hbar^2}\phi \tag{6.4}$$

with associated nonlinear potential

$$\tilde{U}[\rho] = \frac{\beta}{2}\rho^2 + \frac{m\boldsymbol{\alpha}^2}{2\hbar}\rho. \tag{6.5}$$

Equation (6.4) can be reduced to the cubic NLSE by changing the phase $\mathcal{S} \to \mathcal{S} - 3m\boldsymbol{\alpha}^2 t/2\hbar$.

(ii) Let us consider as our second example of canonical NLSE the Chen–Lee–Liu equation [24]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{\alpha}{\hbar}\frac{\partial S}{\partial x}\rho\psi + i\frac{\alpha}{2}\frac{\partial\rho}{\partial x}\psi \tag{6.6}$$

with α a real coupling constant. The nonlinear potential associated with this equation is

$$U[\rho, S] = -\frac{\alpha}{2\hbar}\frac{\partial S}{\partial x}\rho^2 \tag{6.7}$$

while the transformation with generator

$$\sigma = -\frac{\alpha m}{2\hbar}\int\rho\,\mathrm{d}x \tag{6.8}$$

reduces equation (6.6) to the following noncanonical NLSE:

$$i\frac{\partial\phi}{\partial t} = -\frac{\partial^2\phi}{\partial x^2} - \frac{\alpha}{\hbar}\left(\frac{\partial\mathcal{S}}{\partial x} + \frac{3\alpha m}{8\hbar}\rho\right)\rho\phi. \tag{6.9}$$

The transformation with generator σ given by equation (6.8) is a particular case of the Kundu transformation introduced in [25].

(iii) As our third example we consider the canonical NLSE introduced in [26, 27]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{\lambda}{m}\left(\frac{\lambda}{8}\rho + \frac{\partial S}{\partial x}\right)\rho\psi + i\frac{\hbar\lambda}{2m}\frac{\partial\rho}{\partial x}\psi. \tag{6.10}$$

The associated potential is given by

$$U[\rho, S] = -\frac{3\lambda^2}{8m}\rho^3 - \frac{\lambda}{2m}\frac{\partial S}{\partial x}\rho^2 \tag{6.11}$$

while the generator σ assumes the form

$$\sigma = -\frac{\lambda}{2}\int\rho\,\mathrm{d}x. \tag{6.12}$$

The evolution equation for the field ϕ becomes

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} - \frac{\lambda}{m}\frac{\partial\mathcal{S}}{\partial x}\rho\phi. \qquad (6.13)$$

(iv) As our fourth example we consider the canonical NLSE recently introduced in [28, 29]:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\kappa}{m}\rho\left(\frac{\partial S}{\partial x}\right)^2\psi - \mathrm{i}\frac{\kappa\hbar}{2m\rho}\frac{\partial}{\partial x}\left(\rho^2\frac{\partial S}{\partial x}\right)\psi. \qquad (6.14)$$

The nonlinear potential associated with equation (6.14) is given by

$$U[\rho, S] = \frac{\kappa}{2m}\left(\rho\frac{\partial S}{\partial x}\right)^2. \qquad (6.15)$$

Although equation (6.14) can be generalized in any spatial dimension, it is easy to verify that the condition (4.8) is not satisfied in this case; thus we can apply the transformation only to the one-dimensional case. We perform the transformation generated by

$$\sigma = \kappa\int\rho\frac{\partial S}{\partial x}\,\mathrm{d}x \qquad (6.16)$$

and equation (6.14) transforms into

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + \frac{\kappa}{m}\frac{\rho}{1+\kappa\rho}\left(\frac{\partial\mathcal{S}}{\partial x}\right)^2\phi - \kappa\frac{\hbar^2}{4m}\rho\frac{\partial^2\log\rho}{\partial x^2}\phi. \qquad (6.17)$$

(v) As a last example of the canonical system we consider the sub-class of DG equations given by [6]:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \left\{\alpha\Delta S - 2\beta\frac{\hbar^2}{m}\left[\frac{\Delta\rho}{\rho} - \frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2\right]\right\}\psi + \mathrm{i}\alpha\frac{\hbar}{2}\frac{\Delta\rho}{\rho}\psi \qquad (6.18)$$

with associated nonlinear potential

$$U[\rho, S] = \alpha\rho\Delta S + \beta\frac{\hbar^2}{m}\frac{(\nabla\rho)^2}{\rho}. \qquad (6.19)$$

The generator σ is now

$$\sigma = -m\alpha\log\rho \qquad (6.20)$$

while the evolution equation for the field ϕ becomes

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \gamma\left[\frac{\Delta\rho}{\rho} - \frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2\right]\phi \qquad (6.21)$$

with $\gamma = m\alpha^2 - 2\beta\hbar^2/m$. This equation is again canonical, with nonlinear potential

$$\tilde{U}[\rho] = -\frac{\gamma}{2}\frac{(\nabla\rho)^2}{\rho} \qquad (6.22)$$

and can be linearized performing the rescaling $\mathcal{S} \to \mathcal{S}\sqrt{2m\gamma/\hbar^2 - 1}$, as noted in [30].

(vi) The most general class of DG equations is noncanonical and takes the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \hbar D'\sum_{i=1}^{5}c_i R_i[\rho, S]\psi + \mathrm{i}\frac{\hbar}{2}DR_2[\rho, S]\psi \qquad (6.23)$$

where $R_1 = \nabla\cdot\boldsymbol{j}/\rho$, $R_2 = \Delta\rho/\rho$, $R_3 = (\boldsymbol{j}/\rho)^2$, $R_4 = \boldsymbol{j}\cdot\nabla\rho/\rho^2$, $R_5 = (\nabla\rho/\rho)^2$. Note that equation (6.18) is obtained when $D = \alpha$, $c_1 = -c_4 = m\alpha/\hbar D'$, $c_3 = 0$ and

$c_2 = -2c_5 = -2\beta\hbar/mD'$. The same generator σ given by equation (6.20) defines the transformation $\psi \to \phi$ reducing the evolution equation to

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \sum_{i=1}^{5}\tilde{c}_i R_i[\rho, \mathcal{S}]\phi \tag{6.24}$$

where now the coefficients are given by $\tilde{c}_1 = \hbar D'c_1 - mD$, $\tilde{c}_2 = \hbar D'c_2$, $\tilde{c}_3 = \hbar D'c_3$, $\tilde{c}_4 = \hbar D'c_4 + mD$, $\tilde{c}_5 = \hbar D' + mD^2/2$. Note that in [7] a nonlinear transformation was introduced with generator

$$\sigma_{DG} = \frac{\gamma(t)}{2}\log\rho + \frac{1}{\hbar}[\lambda(t) - 1]S + \theta(t, \boldsymbol{x}) \tag{6.25}$$

which produces a group of transformations mapping the DG equation into itself. We observe that, after posing in equation (6.25) $\theta(t, \boldsymbol{x}) = 0$, $\lambda(t) = 1$ and $\gamma(t) = 2m\beta/\hbar$, we obtain the generator σ given by equation (6.20).

(vii) As a second example of noncanonical system we consider the equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \mathrm{i}\alpha\left(\psi^*\frac{\partial\psi}{\partial x} + q\psi\frac{\partial\psi^*}{\partial x}\right)\psi \tag{6.26}$$

with q a real parameter. We note that for $q = 1/2$, equation (6.26) reduces to the Kaup–Newell equation [31], while for $q = 0$ we obtain the Chen–Lee–Liu equation (6.6). Finally, for $q = -1$ the nonlinearity in equation (6.26) becomes purely real and the equation coincides with equation (6.13) obtained previously. The generator

$$\sigma = -\frac{m\alpha}{2\hbar}(q + 1)\int\rho\,\mathrm{d}x \tag{6.27}$$

defines a transformation which reduces equation (6.26) to

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} - \frac{\alpha}{\hbar}\left[(1 - q)\frac{\partial\mathcal{S}}{\partial x} + \frac{1}{8}m\frac{\alpha}{\hbar}(3 - 2q - 5q^2)\rho\right]\rho\phi. \tag{6.28}$$

In the particular case $q = 1$ equation (6.28) becomes canonical with nonlinear potential

$$\tilde{U}[\rho] = \frac{m\alpha^2}{6\hbar^2}\rho^3. \tag{6.29}$$

(viii) As a final example we consider the Eckaus equation [32] which is a noncanonical NLSE:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \mathrm{i}\alpha\frac{\partial\rho}{\partial x}\psi + \beta\rho^2\psi. \tag{6.30}$$

The generator

$$\sigma = -\frac{m\alpha}{\hbar}\int\rho\,\mathrm{d}x \tag{6.31}$$

defines the transformation reducing equation (6.30) to the well known quintic NLSE:

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + \left(\frac{m\alpha^2}{2\hbar^2} + \beta\right)\rho^2\phi. \tag{6.32}$$

Equation (6.32) is a canonical one with nonlinear potential

$$\tilde{U}[\rho] = \frac{1}{3}\left(\frac{m\alpha^2}{2\hbar^2} + \beta\right)\rho^3. \tag{6.33}$$

In the particular case where $\beta = -m\alpha^2/2\hbar^2$ the transformation with generator given by equation (6.31) linearizes equation (6.30).

7. Conclusion

In this paper we have considered a class of canonical NLSEs containing complex nonlinearities and describing $U(1)$-invariant systems. For these systems we studied the symmetries and the conserved quantities associated to roto-translations and Galilei invariance.

Subsequently, we introduced a Cole–Hopf-like transformation $\psi \to \mathcal{U}\psi$, which preserves the $U(1)$-invariance of the system and reduces the complex nonlinearity into a real one so that the continuity equation assumes the standard bilinear form. This transformation generally does not conserve the canonicity of the system. Extension to noncanonical equations was also studied.

The general Cole–Hopf-like transformation introduced here allows us to treat in a unifying scheme several NLSEs already known in the literature, obtaining, in this way, the transformations introduced by various authors.

Appendix A

In this appendix we recover, by using the Noether theorem [23], the continuity equation associated with a given symmetry.

Let us consider the action

$$\mathcal{A} = \int \mathcal{L}\, \mathrm{d}^n x\, \mathrm{d}t \tag{A.1}$$

with Lagrangian

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_{NL} \tag{A.2}$$

where

$$\mathcal{L}_L = \mathrm{i}\frac{\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2 \tag{A.3}$$

is the standard Lagrangian density of the linear Schrödinger theory while

$$\mathcal{L}_{NL} = -U[\rho, S] \tag{A.4}$$

is a real scalar functional depending on the hydrodynamic fields ρ, S and their spatial derivatives. The evolution equation for the field ψ is given by

$$\frac{\partial \mathcal{A}}{\partial \psi^*} = 0. \tag{A.5}$$

Taking the functional derivatives, equation (A.5) becomes

$$\begin{aligned}\frac{\partial \mathcal{L}_L}{\partial \psi^*} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}_L}{\partial(\partial_t \psi^*)} - \frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}_L}{\partial(\partial_i \psi^*)} + \sum_{[k=0]}(-1)^k \mathcal{D}_{I_k}\left[\frac{\partial \mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}\rho)}\right]\psi \\ + \mathrm{i}\frac{\hbar}{2\rho}\sum_{[k=0]}(-1)^k \mathcal{D}_{I_k}\left[\frac{\partial \mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k} S)}\right]\psi = 0.\end{aligned} \tag{A.6}$$

We compute the variation $\delta_\epsilon \mathcal{A}$ generated by a one-parameter transformation group. For simplicity, we assume that the symmetry group acts only on the internal degrees of freedom of the system. The contributions to $\delta_\epsilon \mathcal{A}$, given by the variation of the volume element $\mathrm{d}^n x\, \mathrm{d}t$, when the symmetry involves the space–time variables, are well known and can be added successively. Thus, we have

$$\delta_\epsilon\mathcal{A} = \int\left[\frac{\delta\mathcal{L}_L}{\delta\psi}\delta_\epsilon\psi + \frac{\delta\mathcal{L}_L}{\delta\psi^*}\delta_\epsilon\psi^* + \frac{\delta\mathcal{L}_{NL}}{\delta\rho}\delta_\epsilon\rho + \frac{\delta\mathcal{L}_{NL}}{\delta S}\delta_\epsilon S\right]\mathrm{d}^n x\,\mathrm{d}t$$

$$= \int\left\{\frac{\partial\mathcal{L}_L}{\partial\psi}\delta_\epsilon\psi + \frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi)}\delta_\epsilon(\partial_t\psi) + \frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi)}\delta_\epsilon(\partial_i\psi)\right.$$

$$+\frac{\partial\mathcal{L}_L}{\partial\psi^*}\delta_\epsilon\psi^* + \frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi^*)}\delta_\epsilon(\partial_t\psi^*) + \frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi^*)}\delta_\epsilon(\partial_i\psi^*)$$

$$\left.+\sum_{[k=0]}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}\rho)}\delta_\epsilon(\mathcal{D}_{I_k}\rho) + \frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}S)}\delta_\epsilon(\mathcal{D}_{I_k}S)\right]\right\}\mathrm{d}^n x\,\mathrm{d}t \tag{A.7}$$

with $\mathcal{D}_{I_k} \equiv \partial^k/(\partial x_1^{i_1}\cdots x_n^{i_n})$ and the Einstein convention for the repeated indices is assumed. In equation (A.7) we have posed $\sum_{[k=0]} \equiv \sum_{k=0}^{\infty}\sum_{I_k}$, where the second sum is performed on the multi-index $I_k \equiv (i_1, i_2, \ldots, i_n)$ with $0 \leqslant i_p \leqslant k$, $\sum i_p = k$. If we use the identity

$$\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta_\epsilon(\partial_a\phi) = \frac{\partial}{\partial a}\left[\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta_\epsilon\phi\right] - \frac{\partial}{\partial a}\left[\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\right]\delta_\epsilon\phi \tag{A.8}$$

with $a \equiv t, i$, equation (A.7) becomes

$$\delta_\epsilon\mathcal{A} = \int\left\{\frac{\partial\mathcal{L}_L}{\partial\psi}\delta_\epsilon\psi + \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi)}\delta_\epsilon\psi\right] - \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi)}\right]\delta_\epsilon\psi + \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi)}\delta_\epsilon\psi\right)\right.$$

$$-\frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi)}\right]\delta_\epsilon\psi + \frac{\partial\mathcal{L}_L}{\partial\psi^*}\delta_\epsilon\psi^* + \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi^*)}\delta_\epsilon\psi^*\right]$$

$$-\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi^*)}\right]\delta_\epsilon\psi^* + \frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi^*)}\delta_\epsilon\psi^*\right] - \frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi^*)}\right]\delta_\epsilon\psi^*$$

$$\left.+\sum_{[k=0]}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}\rho)}\delta_\epsilon(\mathcal{D}_{I_k}\rho) + \frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}S)}\delta_\epsilon(\mathcal{D}_{I_k}S)\right]\right\}\mathrm{d}^n x\,\mathrm{d}t. \tag{A.9}$$

For a fixed value of the index k and multi-index I_k, using k times equation (A.8), we have

$$\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}\rho)}\delta_\epsilon(\mathcal{D}_{I_k}\rho) = \sum_{[p=0]}^{k}(-1)^p A_{I_q}^{I_k}\mathcal{D}_{I_q}\left[\mathcal{D}_{I_p}\left(\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}\rho)}\right)\delta_\epsilon\rho\right] \tag{A.10}$$

where the coefficient $A_{I_q}^{I_k} = \prod_{r=1}^n i_r!/(l_r!m_r!)$, $\sum_{[p=0]}^k \equiv \sum_{p=0}^k\sum_{I_p}$ and the multi-indices $I_k = (i_1, \ldots, i_n)$, $I_p = (l_1, \ldots, l_n)$ and $I_q = (m_1, \ldots, m_n)$ are related by $i_r = l_r + m_r$.

Using equation (A.10), equation (A.9) transforms to

$$\delta_\epsilon\mathcal{A} = \int\left\{\frac{\partial\mathcal{L}_L}{\partial\psi}\delta_\epsilon\psi + \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi)}\delta_\epsilon\psi\right] - \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi)}\right]\delta_\epsilon\psi + \frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi)}\delta_\epsilon\psi\right]\right.$$

$$-\frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi)}\right]\delta_\epsilon\psi + \frac{\partial\mathcal{L}_L}{\partial\psi^*}\delta_\epsilon\psi^* + \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi^*)}\delta_\epsilon\psi^*\right]$$

$$-\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi^*)}\right]\delta_\epsilon\psi^* + \frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi^*)}\delta_\epsilon\psi^*\right] - \frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi^*)}\right]\delta_\epsilon\psi^*$$

$$\left.+\sum_{[k=0]}\sum_{[p=0]}^{k}(-1)^p A_{I_q}^{I_k}\mathcal{D}_{I_q}\left\{\mathcal{D}_{I_p}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}\rho)}\right]\delta_\epsilon\rho + \mathcal{D}_{I_p}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}S)}\right]\delta_\epsilon S\right\}\right\}\mathrm{d}^n x\,\mathrm{d}t. \tag{A.11}$$

After inserting in equation (A.11) the expressions of $\partial\mathcal{L}_{NL}/\partial\psi$ and $\partial\mathcal{L}_{NL}/\partial\psi^*$ obtained from equation (A.6) and its conjugate, we finally obtain

$$\delta_\epsilon\mathcal{A} = \int\left\{\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi)}\delta_\epsilon\psi + \frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi^*)}\delta_\epsilon\psi^*\right] + \frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi)}\delta_\epsilon\psi + \frac{\partial\mathcal{L}_L}{\partial(\partial_i\psi^*)}\delta_\epsilon\psi^*\right]\right.$$
$$+\sum_{[k=1]}\sum_{[p=0]}^{k-1}(-1)^p A^{I_k}_{I_q}\mathcal{D}_{I_q}\left\{\mathcal{D}_{I_p}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}\rho)}\right]\delta_\epsilon\rho\right.$$
$$\left.\left.+\mathcal{D}_{I_p}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{I_k}S)}\right]\delta_\epsilon S\right\}\right\}\mathrm{d}^n x\,\mathrm{d}t. \tag{A.12}$$

In the presence of a symmetry the variation of the action must vanish and thus, from equation (A.12), after rearranging the terms, we derive the continuity equation

$$\frac{\partial\mathcal{Q}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = 0 \tag{A.13}$$

with charge

$$\mathcal{Q} = \frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi)}\delta_\epsilon\psi + \frac{\partial\mathcal{L}_L}{\partial(\partial_t\psi^*)}\delta_\epsilon\psi^* \tag{A.14}$$

and flux $\boldsymbol{\mathcal{F}}$

$$\mathcal{F}_j = \frac{\partial\mathcal{L}_L}{\partial(\partial_j\psi)}\delta_\epsilon\psi + \frac{\partial\mathcal{L}_L}{\partial(\partial_j\psi^*)}\delta_\epsilon\psi^* + \sum_{[k=0]}\sum_{[p=0]}^{k}(-1)^p B^{I_k}_{j,I_q}\mathcal{D}_{I_q}$$
$$\times\left\{\mathcal{D}_{I_p}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{j,I_k}\rho)}\right]\delta_\epsilon\rho + \mathcal{D}_{I_p}\left[\frac{\partial\mathcal{L}_{NL}}{\partial(\mathcal{D}_{j,I_k}S)}\right]\delta_\epsilon S\right\} \tag{A.15}$$

where $\mathcal{D}_{j,I_k}a \equiv \partial_j\mathcal{D}_{I_k}a$, with $a \equiv \rho$, S, and the coefficients $B^{I_k}_{j,I_q} = (i_j+1)A^{I_k}_{I_q}/(m_j+1)f^{I_q}_j$ and $f^{I_q}_j = n - \sum^n_{r\neq j}\delta_{0,m_r}$. Recall that from the continuity equation the current is defined modulo the curl of an arbitrary function. This fact was taken into account in the expression of the current (A.15).

In the following we discuss two important cases. In the first, we suppose that the system is $U(1)$-invariant. Using the transformation $\psi \to \psi\exp(\mathrm{i}\epsilon)$ where ϵ is the infinitesimal generator, we have

$$\begin{aligned}&\delta_\epsilon\psi = \mathrm{i}\epsilon\psi \qquad &&\delta_\epsilon\psi^* = -\mathrm{i}\epsilon\psi^*\\ &\delta_\epsilon\rho = 0 &&\delta_\epsilon S = \hbar\epsilon.\end{aligned} \tag{A.16}$$

From equations (A.14) and (A.15) we obtain the expression of the conserved density $\mathcal{Q} = \rho$ and the related current $\mathcal{F}_i = j_i$:

$$j_i = \frac{\partial_i S}{m}\rho + \sum_{[k=0]}\frac{(-1)^k}{f_i^{I_{k+1}}}\mathcal{D}_{I_k}\left[\frac{\partial U[\rho,S]}{\partial(\mathcal{D}_{i,I_k}S)}\right]. \tag{A.17}$$

It is trivial to note that equation (A.13) is the continuity equation for the field ψ, where the current (A.17) assumes a nonstandard expression, due to the presence of the imaginary part of the nonlinearity in the evolution equation. By taking into account the definition (2.4) of the functional derivative, equation (A.17) can be also written in the form

$$j_i = \frac{\partial_i S}{m}\rho + \frac{\delta}{\delta(\partial_i S})\int U[\rho,S]\,\mathrm{d}^n x\,\mathrm{d}t \tag{A.18}$$

modulo a curl of an arbitrary function. Note that equation (A.18) can be obtained directly from equation (2.7) after adopting the hypothesis that $U[\rho, S]$ depends on the field S only through its spatial derivatives as required from $U(1)$ symmetry.

In the second case, we discuss the energy–momentum tensor related to the space–time translations. Posing $x_\mu \to x_\mu + \epsilon_\mu$ we have

$$\begin{aligned} &\delta_\epsilon \psi = \epsilon_\mu \partial_\mu \psi && \delta_\epsilon \psi^* = \epsilon_\mu \partial_\mu \psi^* \\ &\delta_\epsilon \rho = \epsilon_\mu \partial_\mu \rho && \delta_\epsilon S = \epsilon_\mu \partial_\mu S \end{aligned} \tag{A.19}$$

with $\mu = 0, \ldots, 3$ and $\partial_0 \equiv \partial_t$. From equations (A.14) and (A.15) we obtain

$$T_{00} = \frac{\hbar^2}{2m} |\nabla \psi|^2 + U[\rho, S] \tag{A.20}$$

$$T_{0j} = \mathrm{i}\frac{\hbar}{2} (\psi^* \partial_j \psi - \psi \partial_j \psi^*) \tag{A.21}$$

$$\begin{aligned} T_{i0} = &-\frac{\hbar^2}{2m} (\partial_i \psi^* \partial_t \psi - \partial_i \psi \partial_t \psi^*) \\ &+ \sum_{[k=0]} \sum_{[p=0]}^{k} (-1)^p B_{j,I_q}^{I_k} \mathcal{D}_{I_q} \left\{ \mathcal{D}_{I_p} \left[\frac{\partial U[\rho, S]}{\partial \left(\mathcal{D}_{i,I_k} \rho \right)} \right] \partial_t \rho + \mathcal{D}_{I_p} \left[\frac{\partial U[\rho, S]}{\partial \left(\mathcal{D}_{i,I_k} S \right)} \right] \partial_t S \right\} \end{aligned} \tag{A.22}$$

$$\begin{aligned} T_{ij} = &-\frac{\hbar^2}{2m} \left(\partial_i \psi^* \partial_j \psi + \partial_j \psi^* \partial_i \psi \right) + \delta_{ij} \mathcal{L} \\ &- \sum_{[k=0]} \sum_{[p=0]}^{k} (-1)^p B_{j,I_q}^{I_k} \mathcal{D}_{I_q} \left\{ \mathcal{D}_{I_p} \left[\frac{\partial U[\rho, S]}{\partial \left(\mathcal{D}_{i,I_k} \rho \right)} \right] \partial_j \rho + \mathcal{D}_{I_p} \left[\frac{\partial U[\rho, S]}{\partial \left(\mathcal{D}_{i,I_k} S \right)} \right] \partial_j S \right\}. \end{aligned} \tag{A.23}$$

In equations (A.20) and (A.23) we have taken into account the contribution due to the volume element. Note that the potential $U[\rho, S]$ does not modify the expression of the momentum density T_{0j} which assumes the same form as in the linear theory. In contrast, $U[\rho, S]$ changes the expression of the energy density T_{00}, and even more strongly the expression of the flux densities $T_{i\mu}$.

Appendix B

Theorem. *If $U[\rho, S]$ and $\boldsymbol{F}[\rho, S]$ are two smooth functionals depending on the fields ρ, S and their spatial derivatives and satisfy the relation*

$$\frac{\partial}{\partial S} U[\rho, S] = \nabla \cdot \boldsymbol{F}[\rho, S] \tag{B.1}$$

the functional $U[\rho, S]$ takes the form

$$U[\rho, S] = \overline{U}[\rho, S] + \nabla \cdot \boldsymbol{G}[\rho, S] \tag{B.2}$$

where $\overline{U}[\rho, S]$ depends on S only through its derivatives: $\partial \overline{U} / \partial S = 0$.

Proof. Deriving equation (B.2) with respect to S we obtain

$$\begin{aligned} \frac{\partial U}{\partial S} = \frac{\partial \overline{U}}{\partial S} + \frac{\partial}{\partial S} \nabla \cdot \boldsymbol{G} &= \frac{\partial}{\partial S} \sum_{[k=0]} \left[\partial_i (\mathcal{D}_{I_k} \rho) \frac{\partial G_i}{\partial (\mathcal{D}_{I_k} \rho)} + \partial_i (\mathcal{D}_{I_k} S) \frac{\partial G_i}{\partial (\mathcal{D}_{I_k} S)} \right] \\ &= \sum_{[k=0]} \left[\partial_i (\mathcal{D}_{I_k} \rho) \frac{\partial}{\partial (\mathcal{D}_{I_k} \rho)} + \partial_i (\mathcal{D}_{I_k} S) \frac{\partial}{\partial (\mathcal{D}_{I_k} S)} \right] \frac{\partial G_i}{\partial S} = \nabla \cdot \frac{\partial \boldsymbol{G}}{\partial S} \end{aligned} \tag{B.3}$$

which coincides with equation (B.1) for $\boldsymbol{F} = \partial \boldsymbol{G} / \partial S$.

Alternatively, expanding the rhs of equation (B.1),

$$\frac{\partial U}{\partial S} = \sum_{[k=0]} \left[\partial_i(\mathcal{D}_{I_k}\rho) \frac{\partial F_i}{\partial(\mathcal{D}_{I_k}\rho)} + \partial_i(\mathcal{D}_{I_k}S) \frac{\partial F_i}{\partial(\mathcal{D}_{I_k}S)} \right] \tag{B.4}$$

and integrating on the field S, after taking into account that ρ, S, $\mathcal{D}_{I_k}\rho$ and $\mathcal{D}_{I_k}S$ are independent quantities, we have

$$\begin{aligned} U &= \sum_{[k=0]} \int \left[\partial_i(\mathcal{D}_{I_k}\rho) \frac{\partial F_i}{\partial(\mathcal{D}_{I_k}\rho)} + \partial_i(\mathcal{D}_{I_k}S) \frac{\partial F_i}{\partial(\mathcal{D}_{I_k}S)} \right] \mathrm{d}S \\ &= \sum_{[k=0]} \partial_i(\mathcal{D}_{I_k}\rho) \int \frac{\partial F_i}{\partial(\mathcal{D}_{I_k}\rho)} \,\mathrm{d}S + \sum_{[k=1]} \partial_i(\mathcal{D}_{I_k}S) \int \frac{\partial F_i}{\partial(\mathcal{D}_{I_k}S)} \,\mathrm{d}S \\ &\quad + (\partial_i S) \int \frac{\partial F_i}{\partial S} \,\mathrm{d}S \\ &= \sum_{[k=0]} \partial_i(\mathcal{D}_{I_k}\rho) \frac{\partial}{\partial(\mathcal{D}_{I_k}\rho)} \int F_i \,\mathrm{d}S + \sum_{[k=1]} \partial_i(\mathcal{D}_{I_k}S) \frac{\partial}{\partial(\mathcal{D}_{I_k}S)} \int F_i \,\mathrm{d}S \\ &\quad + (F_i + C_i)\partial_i S \\ &= \boldsymbol{\nabla} \cdot \int \boldsymbol{F} \,\mathrm{d}S + \boldsymbol{C} \cdot \boldsymbol{\nabla} S \end{aligned} \tag{B.5}$$

with C_i integration constants not depending on S. Equation (B.5) coincides with equation (B.2) for $\boldsymbol{G} = \int \boldsymbol{F} \,\mathrm{d}S$ and $\overline{U} = \boldsymbol{C} \cdot \boldsymbol{\nabla} S$. □

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